MANICALAND STATE UNIVERSITY OF APPLIED SCIENCES

FACULTY OF APPLIED SCIENCES & TECHNOLOGY

DEPARTMENT OF APPLIED STATISTICS

ASTA 104 - PROBABILITY THEORY I

April 2023

DURATION: 3 hours

INSTRUCTIONS TO CANDIDATE

1. Answer **ALL** Questions in Section A and any **THREE** Questions from Section B.
2. Start each question on a fresh page.
3. Show **All** your working.

SECTION A [40 marks]

Answer **ALL** Questions being careful to number them A1 to A4.

A1. (a) Let $\Omega = (-\infty, +\infty)$ be the universal set. Use DeMorgan's rules to find

$$\{[0,2] \cap (1,4]\}^c$$

. [4]

(b) Let A and B be events and let $A\Delta B = (A \cap B^c) \cup (B \cap A^c)$. Prove that:

(i) $P(A\Delta B) = P(A) + P(B) - 2P(A \cap B)$ [4]

(ii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ [4]

A2. A die is loaded so that the probability of each face coming up is proportional to the face value

$$p(x) = cx, \quad x = 1, 2, ..., 6$$

for some constant c.

(a) Find the value of c. [3]

(b) Find the CDF of X. [4]

(c) Find the probability that X is even. [4]

A3. The number of accidents per week in a certain factory follows a Poisson distribution with variance 3.2. Find the probability that

(a) no accident occur in a particular week. [3]

(b) less than 3 accidents in a particular fortnight. [3]

(c) exactly 7 accidents occur in a particular fortnight [4]

A4. Let X be a continuous random variable with parameter λ and probability density function:

$$f_x(x) = \lambda e^{\lambda x}, \quad x > 0, \quad \lambda > 0$$

(a) Show that $E(X) = \frac{1}{\lambda}$. [3]

(b) Show that $Var(X) = \frac{1}{\lambda^2}$. [4]

SECTION B [60 marks]

Answer any **THREE** Questions being careful to number them B5 to B8.

B5. (a) The weight of students in the Department of Mathematics has been observed over the years that about 20% of the time it has not exceeded 58kg and about 25% of the time it has not been less than 70kg. Suppose that the weight is assumed to be normally distributed, find the mean and standard deviation. [8]

(b) In a certain factory there are two machines producing the same brand of light bulbs. The first machine produces 10% and the second machine produces 90% of the bulbs. Let the probability that the first machine turns out a defective light bulb be 1% and the probability that the second machine turns out a defective light bulb be 5%.

(i) What is the probability that a bulb drawn at random from the production line is defective. [4]

(ii) Given that a bulb is defective, what is the probability that it was produced by first machine. [4]

(iii) Given that a bulb is not defective, find the probability that it was produced by the second machine. [4]

B6. (a) The two events A and B are such that $P(A) = 0.6$, $P(B) = 0.2$, $P(A \mid B) = 0.1$. Calculate the probabilities that:

(i) both of the events occur. [2]

(ii) at least one of the events occur. [2]

(iii) exactly one of the events occurs. [2]

(iv) B occurs, given that A has occurred. [3]

(b) Let X be a uniform continuous random variable over the interval $[a, b]$ with density function:

$$f_x(x) = \frac{1}{b-a}, \ a \leq x \leq b$$

(i) Show that X is a legitimate probability density function. [3]

(ii) Find $E(X)$. [3]

(iii) Determine $Var(X)$. [5]

B7. Let X be random variable with pdf given by:

$$f_X(x) = \begin{cases} c & 0 < x < 5 \\ 0 & otherwise \end{cases}$$

(a) Find c. [3]

(b) Find the $E(X)$ and $Var(X)$. [6]

(c) Find $E(2X + 5)$ and $Var(3X + 2)$. [7]

(d) Let $Y = X^2$, find the pdf of Y , $f_Y(y)$. [4]

B8. (a) State and prove the Chebyshev's Inequality. [7]

(b) Suppose that a random variable has mean 5 and standard deviation 1.2. Use Chebyshev's Inequality to estimate the lower bound of the probability that an outcome lies between 3 and 7. [6]

(c) Let X be a random variable with mean 5 and variance 1. Find the lower bound to the probability $P(\mid X - 5 \mid \leq 2)$ [7]

END OF EXAMINATION PAPER