MANICALAND STATE UNIVERSITY OF APPLIED SCIENCES

FACULTY OF APPLIED SCIENCES & TECHNOLOGY

DEPARTMENT OF APPLIED STATISTICS

MODULE: PROBABILITY THEORY II

CODE: ASTA 124

SESSIONAL EXAMINATIONS

OCTOBER 2023

EXAMINER: MRS S MANDIZVIDZA

INSTRUCTIONS

1. Answer **All** in Section A.
2. Answer **three** questions in Section B.
3. Start a new question on a fresh page.
4. Total marks: 100.

Additional material(s)

- Statistical tables, Non-programmable electronic scientific calculator, List of formulae.

SECTION A [40 MARKS]

Answer **ALL** questions in this section

A 1

Given the Poisson random variable(Y/λ),

$$f(x) = \begin{cases} \dfrac{e^{\lambda}\lambda^{y}}{y!} & \text{for} \quad y = 0, 1, 2, 3, ... \\ 0 & \text{for} \quad otherwise \end{cases}$$

(a) Show that the moment generating function (mgf), (3)

$$M_Y(t) = e^{\lambda(e^t - 1)}$$

(b) Hence find the (3)

1. first moment $E(Y)$.
2. second moment $E(Y^2)$.

(c) Hence show that mean of $Y = \lambda$ and $var(Y) = \lambda$. (2)

A 2

Let $X = (X_1, ..., X_n)^T$ be a random vector and $t = (t_1, t_2, ..., t_m)^T \in \Re^n$.

(a) Define the moment generating function for a random vector X . (2)

Now assume that $X = (X_1, ..., X_n)^T$ are independent random vectors in $\Re^n$ and let $X = (X_1, ..., X_n)^T$.

(b) Prove that the moment generating function of the random vector X (defined as a sum) is given by $M_X(t) = \prod_{(i=1)}^{m} M_{Xi}(t)$. Assume that X is a random vector in $\Re^n$,A is an $m * n$ real matrix and $b \in \Re^m$. (5)

(c) Prove that the moment generating function of a related random vector, $Y = AX + b$ is given at $t \in \Re^m$ by $M_Y(t) = e^{t^T b} M_X(A^T t)$ (5)

A 3

(a) Given that a random variable X has a characteristic function $\phi_X(t)$, Prove that the characteristic function of a related random variable $Y = aX + b$ is given by $\phi_Y(t) = e^{ibt}\phi_X(at)$ where is a complex number such that $i^2 = -1$. (8)

(b) If X and Y are independent random variables with characteristic functions $\phi_X(t), \phi_Y(t)$ respectively, prove that the characteristic function of a random variable $Z = X + Y$ is given by $\phi_Z(t) = \phi_X(t)\phi_Y(t)$ (6)

A 4

The probability density function (p.d.f) of a random variable X which has a Cauchy distribution is given by

$$f(x) = \begin{cases} \dfrac{1}{\pi(1+x)} & \text{for} \quad -\infty < x < 0 \\ 0 & \text{for} \quad otherwise \end{cases}$$

(a) Show that $E(X)$ does not exist. (6)

SECTION B [60 MARKS]

Answer any **THREE** questions in this section

A 5

Suppose that Y follows a standard normal distribution with mean 0 and variance 1 and probability density function, pdf given by

$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}}$ $for -\infty < y < \infty$ Now consider the transformation $U = g(Y) = Y^2$. Find the probability density of U using each of the specified methods.

(a) The transformation (Jacobian) transformation technique. (10)

(b) The Cumulative distribution Function, CDF technique. (10)

(c) The moment generating function, m.g.f technique. (7)

(d) Identify the resulting distribution(s). (3)

A 6

Let the pdf of X_1 and X_2 be given by $f_{X_1,X_2}(x_1,x_2) = 2exp[-(x_1+x_2)]$ Consider two random variables Y_1 and Y_2 be defined in the following manner. $Y_1 = 2X_1$, $Y_2 = X_2 - X_1$

(a) Find the joint density of Y_1 and Y_2 and also prove that Y_1 and Y_2 are independent. (15)

Given that X and Y are independent random variables such that $\sim exp(\frac{1}{2})$, that is each has probability density function $\frac{1}{2}exp^{\frac{-x}{2}}$ for $x > 0$.

(b) find the probability density of $\frac{X-Y}{2}$. (7)

(c) Suppose that $f(x) = \frac{x}{2}$ for $0 \le x < 2$ and $g(y) = 2(1-y)$ for $0 \le y < 1$. Determine the function $y(x)$ which will transform $f(x)$ into $g(y)$ (8)

A 7

Let $Z_1, ..., Z_n$ be independent random vectors such that $Z_i \sim N(0,1)$.

(a) Prove that the multivariate moment generating function, of $Z = (Z_1, ..., Z_n)^T$ is given as $M_Z(t) = e^{(t^T \frac{t}{2})}$ (5)

(b) Hence derive the moment generating function of $X = AZ + \mu$, where A is an $n * n$ real matrix and $\mu \varepsilon \Re^n$. If $X = (X_1, ..., X_n)^T \sim N(\mu, \Sigma)$ where Σ is a non-singular definite defined as $\Sigma = A * A^T$ with joint probability density function $f_X(x) = \frac{1}{\sqrt{(2\pi)^n det\Sigma}} exp\frac{-1}{2}(X - \mu)^T \Sigma^{-1}(X - \mu)$ (6)

In the case for the bivariate case(i.e for $n = 2$), $\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$

$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$ where $\mu_i = E(X_i)$ and $Var(X_i) = \sigma_i^2$ for $i = 1,2$

(c) Prove that the joint probability density of $(X_1, X_2)^T \sim N(\mu, \Sigma)$ is given by (9)

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{(2\pi\rho\sigma_1\sigma_2\sqrt{(1-\rho^2)}} * exp[-\frac{1}{2(1-\rho^2)}][(\frac{(X_1-\mu_1)^2}{\sigma_1^2}) + \frac{(X_2-\mu_2)^2}{(\sigma_2^2)} - \frac{2\rho(X_1-\mu_1)(X_2-\mu_2)}{(\sigma_1\sigma_2)}]$$

Suppose that the time (in minutes) required to serve a customer at a certain store has an exponential distribution with a mean of 3.

(d) What is the probability that the time to serve a customer will exceed 3.75 minutes? (5)

(e) Use the Central Limit Theorem to determine the approximate probability that the total time to serve a random sample of 16 customers will exceed 1 hour (5)

A 8

Consider two random variables Y_1 and Y_2 be defined in the following manner. $Y_1 = 2X_1$ $Y_2 = X_2 - X_1$

(a) Find the joint density of Y_1 and Y_2 and also prove that Y_1 and Y_2 are independent. (10)

(b) find the probability density of $\frac{X-Y}{2}$. (7)

Suppose that $f(x) = \frac{x}{2}$ for $0 \leq x < 2$ and $g(y) = 2(1-y)$ for$0 \leq y < 1$.

(c) Determine the function $y(x)$ which will transform $f(x)$ into $g(y)$ (8)