MANICALAND STATE UNIVERSITY OF APPLIED SCIENCES

FACULTY OF ENGINEERING, APPLIED SCIENCES & TECHNOLOGY

DEPARTMENT OF APPLIED STATISTICS

MODULE: ORDINARY DIFFERENTIAL EQUATIONS

CODE: ASTA 214

SESSIONAL EXAMINATIONS

APRIL 2024

DURATION: 3 HOURS

EXAMINER: D. MHINI

INSTRUCTIONS

1. *Answer **All questions** in Section A*
2. *Answer any **three** questions in Section B.*
3. *Start a new question on a fresh page*
4. *Total marks 100*

***Additional material(s):** Non-programmable electronic scientific calculator*

SECTION A

(*Answer* ***ALL*** *questions from this Section)* ***[40]***

A1. (a) State the condition for a set of function $y_1(x), y_2(x), \dots, y_n(x)$ to be a solution of the homogeneous differential

$$a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_0(x)y = 0$$

Hence write down the general solution of the differential equation.

[3]

(b) Write down the Matlab code for solving the second order equation

$$y''(x) + 8y'(x) + 2y(x) = \cos(x)\,; \;\; y(0) = 0, \qquad y'(0) = 1$$

[3]

(c) Define the following terms:

(i) ordinary differential equation;

(ii) non homogeneous equation;

(iii) exact differential equation;

(iv) homogeneous equation.

[4]

A2. A bacteria culture is known to grow at a rate proportional to the amount present. After one hour, 1000 strands of the bacteria are observed in the culture; and after four hours, 3000 strands are observed. Find:

(a) An expression for the approximate number of strands of the bacteria present in the culture at any time t.

(b) The approximate number of strands of the bacteria originally in the culture.

[4, 6]

A3 (a) In the human body, a virus can have two choices of cells to infect ($CD4$ and $CD8$ cells, as examples). The survival of the virus will then be related to its ability to infect as many cell types as possible. The growth patterns of the cells differ in the presence of infection. Let the number of virions be represented by x, $CD4$ cells by y and $CD8$ by z: Let z grow logistically in the absence of the virus and y exponentially in the absence of the virus. Given that x dies out in the absence of the cells, we attempt to model the interactions by the following system of equations:

$$\frac{dx}{dt} = \alpha xz + \beta xy - \gamma x,$$

$$\frac{dy}{dt} = \delta y - \epsilon xy,$$

$$\frac{dz}{dt} = \mu z(r - z) - \rho xz.$$

Describe fully term by term, the above system clearly stating what each parameter stands for. What can this system be likened to, in an ecosystem?

(b)Solve the following differential equation:

$$(3x^2 + y + 1)dy + (3y^2 + x + 1)dy = 0.$$

[6, 4]

A4. (a) According to Newton's law of cooling, the rate at which a substance cools is proportional to the difference between the temperature of the substance and that of the air. If the temperature of the air is $300K$, the substance cools from $370K$ to $340K$ in 15 minutes.

(i) Form a differential equation to represent the process of cooling;

(ii) Hence, find the time when the temperature will be $310K$.

(b) Use the Inverse D-Operator method to solve the general solution of the differential equation:

$$[D^3 - 2D^2 - 5D + 6]y = (e^{2x} + 3).$$

[1, 5, 4]

SECTION B

(*Answer any* ***THREE*** *questions from this section)* ***[60]***

B5 (a) In the investigation of a homicide or accident death it is often important to estimate the time of death. From experimental observations it is known that, to an accuracy satisfactory in many circumstances, the surface temperature of an object changes at a rate proportional to the difference between the temperature of the object and that of the surrounding environment (the ambient temperature). This is known as Newton's law of cooling. If $\theta(t)$ is the temperature of the object at time t, and T is the constant ambient temperature, show that θ satisfy the linear differential equation

$$\frac{d\theta}{dt} = -k(\theta - T);$$

where $k > 0$ is a constant of proportionality.

Suppose that at time $t = 0$ a corpse is discovered, and that its temperature is measured to be θ_0. Assume that at the time of death the body temperature had the normal value of 98.6^0C and that the above differential equation is valid. Given that the temperature of the corpse is 85^0C when discovered and 74^0C two hours later, and that the ambient temperature is 68^0C, determine the time of death, t_d .

(b) Use Frobenius method to find a series solution for the equation

$$2x\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$$

[2, 8, 10]

B6. (a) Solve the system of differential equations:

$$\frac{dx}{dt} = 2x + 3y$$

$$\frac{dy}{dt} = 2x + y$$

subject to: $x(0) = -4; y(0) = -1$

(b)Determine whether the differential equation

$$(2x^2t - 2x^3)dt + (4x^3 - 6x^2t + 2xt^2)dx = 0$$

is exact. Hence solve the differential equation.

(c) Use the method of undetermined coefficients to solve the equation;

$$y''' - 6y'' + 11y' - 6y = 2xe^{-x}$$

[`8, 5, 7]

B7. (a) Solve the following system:

$$\frac{2dx}{dt} + \frac{dy}{dt} - 4xy - y = e^t$$

$$\frac{dy}{dt} + 3x + y = 0$$

(b)Consider the Legendre equation

$$(1 - x^2)y'' - 2xy + k(k+1)y = 0;$$

for any positive integer k

(i) Determine whether $x = 0$ or $x = 1$ is an ordinary point of Legendre equation.
$(1 - x^2)y'' - 2xy + k(k+1)y = 0;$

(ii) Find the recurrence formula for the power series solution around $x = 0$ for the Legendre equation.

(iii) Show that whenever k is a positive integer, one solution near $x = 0$ of Legendre's equation is a polynomial of degree k.

[8, 2, 5, 5]

B8 (a) Use Laplace transforms to solve the differential equation.

$$y'' - 3y' + 2y = 0\, ; y(0) = -3\, , y'(0) = 5.$$

(b)State without proof the existence and uniqueness of solutions theorems.

(c)Find the general solution of the differential equation using the variation of parameters method:

$$x^5y'' + 6x^5y' + 9x^5y = e^{-3x}.$$

[8, 2, 10]

END OF EXAMINATION PAPER