MANICALAND STATE UNIVERSITY OF APPLIED SCIENCES

FACULTY OF APPLIED SCIENCES & TECHNOLOGY

DEPARTMENT OF APPLIED STATISTICS

MODULE: ESTIMATION TECHNIQUES

CODE: HAST225

SESSIONAL EXAMINATIONS

OCTOBER 2021

DURATION: 3 HOURS

EXAMINER: MR A. CHAKAIPA

INSTRUCTIONS

1. *Answer* ***All*** *in Section A*
2. *Answer* ***two*** *questions in Section B.*
3. *Start a new question on a fresh page*
4. *Total marks 100*

Additional material(s): *Non-programmable electronic scientific calculator, Statistical tables.*

SECTION A: *(ANSWER ALL QUESTIONS)* [40 MARKS]

A1.

(a). Define Mean Squared Error (MSE) consistency.

(b). Let $X_1, X_2, \cdots, X_n$ be a random sample obtained from a population whose mean is μ and variance σ^2. Let $a_1, \cdots, a_n$ be weights or constants such that
$\sum_{j=1}^{n} a_j = 1$.
Determine whether or not the following are mean-squared error consistent.

i. $\{U_n\}$ where

$$U_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

ii. $\{V_n\}$ where

$$V_n = \frac{1}{n}\sum_{j=1}^{n} a_j X_j.$$

[2, 3, 5]

A2.

(a). Give the difference between classical approaches (such as Maximum Likelihood estimation) and the Bayesian approaches for estimating the parameter θ

(b). Let $X_1, X_2, \ldots, X_n$ be a random sample from a Bernoulli probability mass function (p.m.f) given by $f(x|\theta) = \theta^x (1-\theta)^{1-x}$, $x = 0,1$
Assume a uniform prior distribution for θ is

$$p_\theta(\theta) = 1, \theta \in (0,1).$$

Find the Posterior Bayes estimator of θ.

[4, 6]

A3.

Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson distribution with parameter μ . Find the Maximum likelihood estimator of parameter

$$\beta = \tau(\mu) = \mu^2.$$

[6]

A4.

Let $X_1, X_2, \ldots, X_n$ be a random sample from an exponential distribution with parameter θ.

(a). Find a lower bound of the variance of unbiased estimator of
$\tau(\theta) = \frac{1}{\theta} = E(X)$.

(b). Is $\bar{X}$ a Uniformly Minimum Variance Unbiased Estimator (UMVUE) of θ.

[10, 4]

SECTION B: *(ANSWER ANY TWO (2) QUESTIONS)* [60 MARKS]

B5.

(a). Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with density function $f(x,\theta) = (1-\theta)x^{-\theta}$ for $0 < x < 1$. What is the maximum likelihood estimator of θ.

(b). Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with density function $f(x,\theta) = \sqrt{2/\Pi}\, exp\{\frac{-1}{2}(x-\theta)^2\}$ for $x > \theta$

What is the maximum likelihood estimator of θ. In your working justify all steps and specify the parameter space of θ.

(c). Suppose that $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with
density

$$f(x,\alpha,\beta) = \frac{1}{\alpha-\beta},$$

for $\alpha < x < \beta$

i. Find the estimators of α and β using the method of likelihood.

ii. State the invariance property of maximum likelihood estimators and hence find the MLE of $\sqrt{\alpha^2+\beta^2}$.

[10, 8, 9, 3]

B6.

(a). A quality controller with an electronic company assigns values to a random variable X as follows

$$X = \begin{cases} 1 & \text{if item is defective} \\ 0 & \text{otherwise} \end{cases}$$

Let $X_1, X_2, \ldots, X_n$ be a random sample obtained from an inspection for defectives of electronic components produced by a machine that is sensitive to operating temperature and other unspecified physical conditions. Let θ be the probability of getting a defective component. The Bernoulli distribution with parameter θ models the X_i's and

$$f_{X_i}(x_i,\theta) = \theta^x(1-\theta)^{1-x}, \quad \theta \in (0,1)$$

The joint distribution of the random variables $X_1, X_2, \ldots, X_n$ would be given by the likelihood function

$$f_{X_1,X_2,\ldots,X_n}(x_1,x_2,\cdots,x_n,\theta) = \prod_{i=1}^{n} f_{Xi}(x_i,\theta), \text{for } x = 0,1$$

where θ is a random variable owing its dependence on the stochastic physical conditions. One possible prior distribution is the Beta distribution, $p_\theta(\theta) = 6\theta\,(1 - \theta)$ and $\theta \in (0,1)$.

Find the Posterior Bayes estimator of θ if the latter prior distribution holds.

(b). Let $X_1, X_2, \ldots, X_n$ be a random sample from a Bernoulli probability mass function (p.m.f) given by $f(x|\theta) = \theta^x(1-\theta)^{1-x}, x = 0,1$. Assume a uniform prior distribution for θ is $p_\theta(\theta) = 1, \theta \in (0,1)$. Find the Posterior Bayes estimator of $\theta(1-\theta)$.

(c). In estimation techniques one needs to select an estimator with properties that are considered (similar to an UMVUE in statistical classical approach). An exploration of a loss function and an associated risk function are used in conjunction with a prior distribution of a parameter Ω in the selection of an optimum estimator. An estimator called the Bayes risk estimator is considered as a Bayesian estimator.

- **i.** Define a Bayesian Risk function.
- **ii.** State any two risk loss functions.

(d). Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed (i.i.d) with pdf $f(x)$ and cumulative distribution function CDF, $F(x)$. Let

$$Y_1 = X_{(1)} = \text{ smallest } (X_1, X_2, \ldots, X_n)$$

Given that the Cumulative Distribution Function (CDF) of Y_n is given by

$$F_{y1}(y) = 1 - [1 - F_x(y)]^n$$

Let the random variable X denote the length of time it takes to complete a Statistics Assignment. Suppose the density function of X is given by

$$f_X(x, \theta) = e^{-(x-\theta)},$$

for $x > \theta$.

- **i.** Show that Y_1 has density function given by $f_{y1}(y) = ne^{-n(y-\theta)}$ for $y > \theta$
- **ii.** Calculate the mean of Y_1

[10, 3, 2, 10, 5]

B7.

(a). Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a distribution with density function

$$f(x, \theta) = 3\theta\, x^2\, exp(-\theta\, x^3),$$

for $0 < x < \infty$. One way to find a Minimum Variance Unbiased Estimator is the Cramer-Rao Lower Bound (CRLB) inequality.

- **i.** State the CRLB inequality.
- **ii.** Find a lower bound of the variance of unbiased estimator of parameter θ.

(b). In many situations, we cannot easily find the distribution of the estimator $\hat{\theta}$ of a parameter θ even though we know the distribution of the population. One criteria employed to know whether an estimator is biased or unbiased is sufficiency. A group of sufficient statistics is often employed for members of the exponential class.

i. State the k-parameter exponential family

ii. Let $X_1, X_2, \ldots, X_n$ be a random sample obtained from a log-normal distribution with parameters μ and σ^2 .

$$f(x, \mu, \sigma^2) = \frac{1}{x\sigma\sqrt{2\pi}} exp\{\frac{-1}{2} (\frac{\ln x - \mu}{\sigma})^2\} I_{(0,\infty)}(x)$$

Find a set of jointly minimal sufficient statistics of μ and σ^2 .

[3, 12, 3, 12]

END OF QUESTION PAPER